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Langton's flies

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Abstract
Langton's ant is an automaton defined over a two-dimensional grid. Its movement is governed by the environment in a very simple way: it turns to the left over white vertices and to the right over black vertices. This definition only applies on a two-dimensional space. We look for generalizations of this automaton to n-dimensional lattices. Remembering the different ways through which the ant was originally defined, we consider two approaches: the first comes from physics (lattice gas) and the second from artificial life (virtual ants). Two generalizations are proposed defining two families of dynamical systems. From the physics point of view, the ant is seen as a *particle* and hence it has no internal state other than its velocity. From the artificial life point of view, the ant is viewed as an *insect*, and it has an orientation in space which we represent by an orthogonal basis. This constitutes the ant's internal state. This formulation allows us to define the ant's behaviour without drawing upon any information relative to the global system of external coordinates. Each model yields different sets of rules with distinctive behaviours. We characterize all the possible rules satisfying some basic restrictions. We found that many rules produce trajectories which are restricted to a diagonal plane and are equivalent to a version of Langton's ant over a two-dimensional grid, squared or hexagonal. In the particle model, only two of them use the whole space, and it is shown that such rules do not admit periodical trajectories. This result reinforces a previous one reported by Leonid Bunimovich, who states that 'the skeleton of any bounded trajectory cannot contain any three-dimensional polyhedron'.

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(Some figures in this article are in colour only in the electronic version)

Introduction

In the 1980s, Christopher Langton defined several discrete systems modelling several aspects of living beings [1]. His famous ant is one of these models. It is defined as an automaton moving on a square planar grid whose vertices are coloured either in black or in white. When the ant enters a vertex, it decides the direction in which it will move according to the colour of the vertex: it turns to its left if the vertex is black and to its right otherwise, and after that, it flips the colour of the vertex. When the ant starts moving on a grid whose vertices are all in the same colour, it has an apparently 'chaotic' behaviour for about 10 000 steps, and then it falls into a periodical movement with a drift. This unexpected behaviour is what Langton got interested in.

But the same system was also studied within physics. In 1985, Gunn and Ortuño [2] defined a discrete version of Lorentz's gas. In their model, the vertices of a two-dimensional grid are supporting scatterers that change the particle's velocity in a right angle. They remarked that changing the ratio of left to right scatterers changes the system's behaviour, and that the problem can be seen as a new type of bond percolation. In 1999, Quas [3] studied the problem for mirror-like scatterers. He proved results about existence and geometry of infinite trajectories and its dependence on the mirrors' concentration. These models are called static models (SM), because the scatterers do not flip. One of the main open problems concerning SM is what happens if the scatterers are rotators instead of mirrors.

Cohen took the Gunn model and gave to the particle the ability to change the scatterers' state. In this way, he added to the model a feedback from the environment to the particle. This modification defines a very different system, more similar for example to self-avoiding paths. The system's behaviour is very different and, through numerical experiments, Cohen showed that the diffusion is abnormal.

Cohen considered the triangular and hexagonal grids, and also some random networks like Delaunay networks. The behaviour of the particle proved to be very sensitive to the grid structure and to the type of the scatterers. For example, on the triangular grid he obtained a propagative movement, even over random initial colorations. This behaviour was finally explained by Grosfils *et al* [4].

On the square grid, the system was intensively studied by Bunimovich and Troubetzkoy [5–8]. Two of their most important results are: (1) in the full-filled model (where every vertex has a scatterer), no trajectory is finite and (2) if some trajectory is bounded in a given model, then each scatterer's rule either has a backward fragment or is a rotator. This second result says that if some of the scatterers' rules is not a rotator and does not contain a backward fragment, then no finite trajectory is possible. Questions such as 'What is the mean length of a finite trajectory?' remain open.

At this point, we have described a system where an object (particle or ant) moves on a grid and finds scatterers (symbols) which make it change its direction and which are modified after the object's visit. This corresponds to a Turing machine description. The difference is that Turing machines are defined to accomplish tasks, their scatterer's rules not being simple, and they have a large set of internal states which are not reduced to physical parameters as velocity. Langton's ant fits the definition of a Turing machine, because we can assimilate its velocity as an internal state, but this state is affected by the environment in an isotropic way. In [9], Turk defined a particular kind of two-dimensional Turing machines, described as an arrow and whose movement instructions were *go ahead, turn left*, etc. With this definition, it seems that less states are needed to make similar computations, as an example, we recall that Langton's ant was proved to be Turing universal in [10]. A similar model was also considered by Brady [11] when studying the busy beaver problem in dimension 2, which consists in finding the halting Turing machine that writes the largest quantity of 1s when it starts over a white tape, and has a small given number of states.

For artificial life, this is a more realistic model because animals generally have their own coordinate system, and have usually no access to a global coordinate system.

In this study, we look for generalizations of Langton's ant to the three-dimensional lattice. We consider the two approaches, from physics and from computer science. In dimension 2, they both lead to systems equivalent to Langton's ant, whereas in dimension 3 (or higher), they define very different systems.

From the physicists' point of view, the ant is described by its position and its velocity. Scatterers that affect the ant's velocity are put on the lattice sites. Their action is simply described by a map on the set of velocities. Such an approach was taken by Bunimovich in [12]. The only difference between Bunimovich's model and ours is that he describes the ant by the lattice edge where it lands, but both models define equivalent dynamical systems. We comment his main result in section 2.2. We consider three restrictions on the transition rule, based on the physical properties of particles: (1) it has to be reversible, (2) it must respect the time reversibility symmetry and (3) it must always change the ant's velocity direction. These conditions lead to only six different rules, we study all of them and prove several properties.

From the computer scientists' approach, the ant is described by its position and a basis of orthogonal vectors which forms its relative coordinate system. At each step, the ant moves in the direction of a preferential axis called 'velocity'. Its coordinate system is then modified by a function associated with the symbol that is on the lattice site where the ant lands. A first experimental study of this kind of model has already been done by Hamann [13]. In his model, the ant was described by its velocity together with an oriented 'plane' that must be parallel to the velocity. At each step, a rotation was made either to the *right*, to the *left*, *up* or *down*. Our approach is more general because it allows any kind of rotation of the coordinate system and it works on arbitrary dimensions. We focus on a particular class of rules whose behaviour can be well characterized in any dimension. Within this class, we obtain only three rules in 3D, and we study each of them.

An important analysis tool we defined is the notion of *support graph*, which consists in the set of vertices and edges that the ant may visit for a given rule. We prove that, in most cases, the support graph is not the whole lattice, and this fact puts in evidence many restrictions of the ant's behaviour.

In section 1, we introduce the formal definitions for the study. Then, in section 2, we explore the physicists' point of view. Section 3 is devoted to develop the computer scientists' approach. Finally, in section 4, we propose some concluding remarks.

1. Definitions

In this section, we give the general definition of the dynamical system, together with some other useful definitions. The main difference between the physicists' and the computer scientists' approaches is on the set of internal states Q, which will be defined in sections 2 and 3.

Both systems are defined on the n-dimensional square grid, which can be viewed as an infinite graph $G = (V, E)$ where $V = \mathbb{Z}^n$ and E connects vertices at distance exactly 1. The ant's velocity is chosen from the set of unitary vectors which are parallel to the edges, i.e., from the set $\mathcal{D} = \{\pm e_i\}_{i=1}^n$, where e_i is the vector whose coordinates are all equal to 0 except for the ith coordinate equal to 1.

A colour is associated to each vertex by a function $c : \mathbb{Z}^n \to C$ where C is a finite set called *colour set*. We will call this function a *colouration*.

Definition 1 (the dynamical system)**.** *An ant is specified by its position on the grid, $x \in \mathbb{Z}^n$, and its internal state $q \in Q$. The state of the system is given by the 3-tuple: (c, x, q), where $c : \mathbb{Z}^n \to C$ is a colouration. The phase space is then: $X = C^{\mathbb{Z}^n} \times \mathbb{Z}^n \times Q$. The dynamics is governed by a global transition function: $F : X \to X$.*

The following concept will be useful to describe the important properties of the ant.

Definition 2 (support graph)**.** *Given an initial state (c, x_0, q) of the system, we define the trace of the ant by the series of positions $(x_i)_{i\in\mathbb{N}}$ reached by the ant during the whole evolution of the system. Let us denote the trace by $\mathcal{T}(c, x_0, q) = (x_i)_{i\in\mathbb{N}}$.*

Given an initial position x_0 and an initial state q, we define the support by the following set of vertices:

$$S(x_0, q) = \{x_j | (\exists c \in C^{\mathbb{Z}^n})(x_i)_{i\in\mathbb{N}} = \mathcal{T}(c, x_0, q) \wedge j \in \mathbb{N}\}.$$

The support graph is the directed graph (S, E), where E is the set of arcs the ant traverses,

$$E(x_0, q) = \{(x_j, x_{j+1}) | (\exists c \in C^{\mathbb{Z}^n})(x_i)_{i\in\mathbb{N}} = \mathcal{T}(c, x_0, q) \wedge j \in \mathbb{N}\}.$$

Most of the present work is a study of support graph, for it can give many details about the movement of the ant. For example, if there exists a sequence of moves for some colouration of the grid that have the ant come back on the same edge with a different direction, then for any arc (x, x') in E, the opposite arc (x', x) is very likely to be also in the support graph. In this case, we will represent the support graph as an undirected graph, while in the other case (which is also the case of Langton's ant), the support graph is a directed graph (digraph) whose arcs are represented by arrows.

Frequently, two systems have a similar behaviour in the sense that they induce the same set of traces. We formalize this concept as follows.

Definition 3 (equivalent rules)**.** *Two transition functions $F_1 : X_1 \to X_1$ and $F_2 : X_2 \to X_2$ are said to be equivalent if there exists a function $\Psi : X_1 \to X_2$ such that the trace of every (c, x, q) under the function F_1, $\mathcal{T}_1(c, x, q)$, is equal to the trace of $\Psi(c, x, q)$ under the function F_2, $\mathcal{T}_2(\Psi(c, x, q))$.*

2. The physicists' approach

From the physicists' point of view, the ant is considered as a particle travelling in a 'forest' of scatterers. It is used as a model for Lorentz's lattice gas. The state of the ant is completely given by its position and velocity, and, therefore, Q is to be the set of velocities. Since we are in a lattice, this set is restricted to the set of canonical directions: $\mathcal{D}$. A scattering rule should be associated with each lattice point. This rule changes the ant's velocity and flips afterwards. With all this, we are ready to define the transition function.

Definition 4 (the ant of physicists)**.** *We call a scattering rule an application $\phi : \mathcal{D} \to \mathcal{D}$. A couple $(e, \phi(e))$ is called a fragment of this rule.*

Let the colour set be $C = \{0, 1, \ldots, k-1\}$. A transition rule is a sequence $(\phi_i)_{i\in C}$ of scattering rules associated with the colours in C.

The state set is the set of velocities $Q = \mathcal{D}$.

The dynamics of the ant transforms a state (c, x, v) of the system into another state $F(c, x, v) = (c', x', v')$ where

- $x' = x + v$,

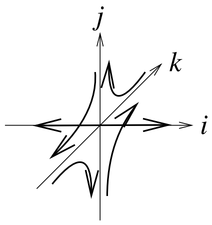
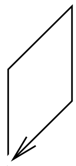
(*a*)  (*b*) 

Figure 1. (*a*) A diagram representing the scattering rule ϕ_0. (*b*) The ant's trace during the first four steps.

- $\boldsymbol{v}' = \phi_{c(x')}(\boldsymbol{v})$,
- $c'(x') = c(x') + 1 \bmod k$,
- $\forall y \neq x', c'(y) = c(y)$.

An example of a scattering rule is $\phi_0 : \boldsymbol{i} \to \boldsymbol{i}, -\boldsymbol{i} \to -\boldsymbol{i}, \boldsymbol{j} \to \boldsymbol{k} \to -\boldsymbol{j} \to -\boldsymbol{k} \to \boldsymbol{j}$ (see figure 1). If the ant starts at $x = (0, 0, 0)$ with velocity $\boldsymbol{v} = \boldsymbol{j}$, then its itinerary in the next four steps will be (0, 1, 0), (1, 1, 0), (1 ,0, 0), (0, 0, 0). At time 4, the ant is back at the origin whose colour is now 1, and the scattering rule ϕ_1 is to be applied.

An important concept, common to many physical systems is *time symmetry*. Informally, this concept says that the inverse dynamics is described by the same transition rule. Formally, a transition function F is time symmetric if for every c, x and $\boldsymbol{v}$ the following implication holds: $F(c, x, \boldsymbol{v}) = (c', x', \boldsymbol{v}') \Rightarrow F(c', x', -\boldsymbol{v}') = (c, x, -\boldsymbol{v})$. Which, in terms of the transition rule is equivalent to

$$\forall(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{D}^2, \qquad \phi_i(\boldsymbol{u}) = \boldsymbol{v} \Rightarrow \phi_{i+1}(-\boldsymbol{v}) = -\boldsymbol{u}.$$

If a transition rule $(\phi_i)_{i\in C}$ is time symmetric, then $\phi_i = \phi_{i+2}$, i.e. the transition rule consists of only two scattering rules which alternate. Moreover, one of the rules is completely determined by the other one. Both rules need to be one to one, hence a time symmetric transformation is reversible. We restrict our study only to time symmetric rules.

In the first subsection, we present an exhaustive study of rules in dimension 3 satisfying time symmetry and some additional geometrical restrictions. In the second subsection, we give some comments and results concerning higher dimensions and more general rules.

2.1. Dimension 3

Here we restrict our attention to time symmetric rules that contain neither straight nor bouncing fragments (of the form $(\boldsymbol{v}, \boldsymbol{v})$ and $(\boldsymbol{v}, -\boldsymbol{v})$ respectively). In this way, we study only the transition rules that change the particle direction in a right angle. For convenience, we will denote by $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ an orthogonal basis of $\mathbb{Z}^3$.

A time symmetric rule is described by a single scattering rule, which needs to be a permutation in $\mathcal{D}$. It is possible to classify the permutations by the length of its basic cycle decomposition.

2.1.1. A symmetric cycle of length 6. The first rule is described by the following diagram:

$$\begin{cases} h_0 : & \boldsymbol{i} \to \boldsymbol{j} \to \boldsymbol{k} \to -\boldsymbol{i} \to -\boldsymbol{j} \to -\boldsymbol{k} \to \boldsymbol{i} \\ h_1 : & \boldsymbol{i} \to -\boldsymbol{k} \to -\boldsymbol{j} \to -\boldsymbol{i} \to \boldsymbol{k} \to \boldsymbol{j} \to \boldsymbol{i}. \end{cases}$$

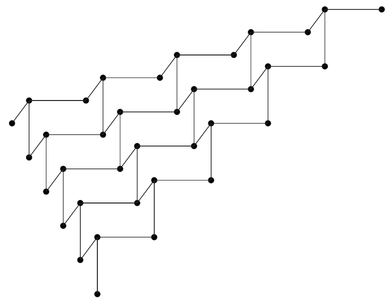


Figure 2. A part of the support graph of the first rule. If the ant starts on an edge of this graph, it will always remain on it.

Similar to Langton's ant's rule, this rule is linear; it is a rotation described by a matrix. The matrices which describe this rule (M_0) and Langton's ant's rule (A_0) have a similar structure,

$$M_0 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad A_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We prove that the ant remains between two diagonal planes such that the cubic grid restricted to these planes is equal to the two-dimensional hexagonal grid. A particular generalization of Langton's ant to the hexagonal grid was already studied by Gajardo in [14]. We prove that the present rule of the ant is equivalent to the one previously studied.

Lemma 1. *If* $(c, x, \boldsymbol{v}) = F^n(c_0, x_0, \boldsymbol{v_0})$ *then*

$$\left(x + \tfrac{1}{2}\boldsymbol{v}\right) \cdot \boldsymbol{a} = C_0, \qquad \textit{where} \quad \boldsymbol{a} = (1, -1, 1)$$

with $C_0 = \left(x_0 + \frac{1}{2}\boldsymbol{v_0}\right) \cdot \boldsymbol{a}$*. In this case, the support is* $S = \left\{x | x \cdot \boldsymbol{a} = C_0 \pm \frac{1}{2}\right\}$.

Proof. Let us remark first that, following the transition rule, the velocity alternates between two sets: $\{\boldsymbol{i}, -\boldsymbol{j}, \boldsymbol{k}\}$ and $\{-\boldsymbol{i}, \boldsymbol{j}, -\boldsymbol{k}\}$, independently on the cell state. This implies that the positions alternate between two planes: $V_i = \left\{x | x \cdot \boldsymbol{a} = C_0 + (-1)^i \frac{1}{2}\right\}$, with $i \in \{0, 1\}$. In fact, let us consider $x_0 \in V_0$ and $\boldsymbol{v_0} \in \{-\boldsymbol{i}, \boldsymbol{j}, -\boldsymbol{k}\}$, if $(c_1, x_1, \boldsymbol{v_1}) = F(c_0, x_0, \boldsymbol{v_0})$, then $\boldsymbol{v_1} \in \{\boldsymbol{i}, -\boldsymbol{j}, \boldsymbol{k}\}$, and $x_1 = x_0 + \boldsymbol{v_0}$. Then $x_1 \cdot \boldsymbol{a} = x_0 \cdot \boldsymbol{a} + \boldsymbol{v_0} \cdot \boldsymbol{a} = C_0 - \frac{1}{2}$, which proves that $x_1 \in V_1$. The lemma is obtained by induction. □

The support graph of the ant is the subgraph of $\mathbb{Z}^3$ generated by S. Figure 2 shows a finite part of this graph. It is not difficult to see that the support graph is a hexagonal grid. The transition rule coincides with the definition given in [14, 15] for the hexagonal grid. To prove it, one just needs to verify if, taking the normal vector a as the orientation of the planes V_0 and V_1, the rule corresponds exactly to turning to the left on colour 0 and to the right on colour 1, i.e., for all $\boldsymbol{v}$, $(\boldsymbol{v} \wedge \phi_0(\boldsymbol{v})) \cdot \boldsymbol{a} > 0$ and $(\boldsymbol{v} \wedge \phi_1(\boldsymbol{v})) \cdot \boldsymbol{a} < 0$.

Now we conclude with the theorem.

Theorem 1. *The ant with transformation rule* (h_0, h_1)*, restricted to its support graph, is isomorphic to the two-dimensional Langton's ant over the hexagonal grid.*

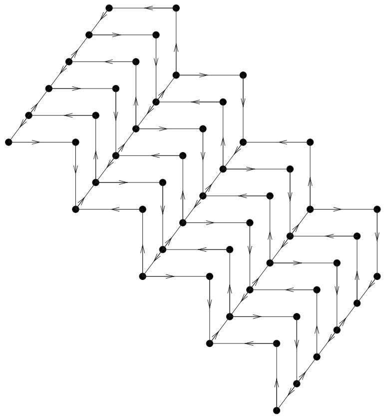


Figure 3. A part of the support graph of the second rule. If the ant starts on an arc of this digraph, it will always remain on it.

2.1.2. An antisymmetric cycle of length 6. The other permutation of length 6 is

$$\begin{cases} l_0: & \boldsymbol{i} \to \boldsymbol{j} \to \boldsymbol{k} \to -\boldsymbol{i} \to -\boldsymbol{k} \to -\boldsymbol{j} \to \boldsymbol{i} \\ l_1: & -\boldsymbol{i} \to \boldsymbol{j} \to \boldsymbol{k} \to \boldsymbol{i} \to -\boldsymbol{k} \to -\boldsymbol{j} \to -\boldsymbol{i}. \end{cases}$$

Let us remark that both l_0 and l_1 contain the fragments $\boldsymbol{j} \to \boldsymbol{k}$ and $-\boldsymbol{k} \to -\boldsymbol{j}$. This means that the particle always takes $\boldsymbol{k}$ after taking $\boldsymbol{j}$ and it takes $-\boldsymbol{j}$ after taking $-\boldsymbol{k}$. Thus we can say that the possible velocities are in fact $\{\boldsymbol{i}, -\boldsymbol{i}, (\boldsymbol{j}+\boldsymbol{k}), (-\boldsymbol{j}-\boldsymbol{k})\}$ rather than $\mathcal{D}$. Hence, the particle stays inside the diagonal plane generated by $\{\boldsymbol{i}, \boldsymbol{j}+\boldsymbol{k}\}$ (see figure 3). Moreover, the rule is isomorphic to the rule of the two-dimensional Langton's ant over the square grid.

2.1.3. A cycle of length 4. Every permutation with a cycle of length 4 results to be equivalent to the following one through a coordinate transformation:

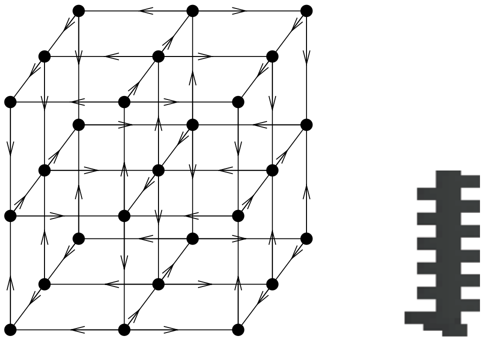
$$\begin{cases} t_0: & \boldsymbol{i} \leftrightarrow \boldsymbol{j} & \text{and} & \boldsymbol{k} \to -\boldsymbol{j} \to -\boldsymbol{k} \to -\boldsymbol{i} \to \boldsymbol{k} \\ t_1: & -\boldsymbol{i} \leftrightarrow -\boldsymbol{j} & \text{and} & \boldsymbol{i} \to \boldsymbol{k} \to \boldsymbol{j} \to -\boldsymbol{k} \to \boldsymbol{i}. \end{cases}$$

This is the most interesting rule of this section. It is not restricted to a two-dimensional part of the space as the former rules. Its support graph contains all the vertices of $\mathbb{Z}^3$ but the edges are directed (see figure 4(a)), i.e. they can be taken in only one direction. Experimentally, we observe that for an initially homogeneous colouration, the ant follows a simple zig-zag movement. But for some periodic colouration, the ant has an initially disordered behaviour, then falling into a regular movement with drift as Langton's ant does (see figure 4(b)).

Lemma 2. *The support graph of this rule, when it starts at position 0 with velocity* $\boldsymbol{j}$, *is* $(\mathbb{Z}^3, E)$, *where E is defined by* $(x = (x_1, x_2, x_3), y) \in E$ *if and only if*

$$\begin{cases} x_1 + x_2 \equiv 0 \pmod 2 & \textit{and} & (y-x) \in \{\boldsymbol{j}, -\boldsymbol{j}, \boldsymbol{k}\} \\ \textit{or} & & \\ x_1 + x_2 \equiv 1 \pmod 2 & \textit{and} & (y-x) \in \{\boldsymbol{i}, -\boldsymbol{i}, -\boldsymbol{k}\}. \end{cases}$$

This proposition is easily proved by induction. We remark that the support graph is directed, i.e., if $(u, v) \in E$, then $(v, u) \notin E$. This allows us to prove that this system has not



(a)fl (b)fl

Figure 4. (a) A part of the support graph of the third rule. If the ant starts on one of its arcs, it will always remain on this digraph. (b) The set of visited cells after 1251 iterations, when starting with a given periodic colouration. This figure shows a projection of the set of visited cells through the x-axis. The movement becomes regular after 786 steps, and each period consists of 104 steps (exactly the same as Langton's ant!). The figure shows five repetitions of the period, the ant advancing four cells upwards at each period.

periodic points. The analogous property for the 2D Langton's ant was proved in [7]. We use a similar technique in our following theorem.

Theorem 2. *For any initial state,* $(c, x, \boldsymbol{v})$*, the trace of the ant* $\mathcal{T}(c, x, \boldsymbol{v})$ *is formed by an infinite number of cells.*

Proof. Let us suppose that the trace $\mathcal{T}$ has a finite number of cells. Then each vertex of $\mathcal{T}$ is visited an infinite number of times. Let us consider the lexicographical order in $\mathbb{Z}^3$ and let us choose a maximal vertex $s = (x_1, x_2, x_3)$ of $\mathcal{T}$. If $x_1 + x_2$ is even, lemma 2 implies that the only way to exit s without exiting $\mathcal{T}$ is with velocity $-\boldsymbol{j}$, then the ant alternately enters s with velocities $-\boldsymbol{i}$ and $\boldsymbol{k}$. But $s + \boldsymbol{i}$ is outside $\mathcal{T}$, which is a contradiction. If we suppose that $x_1 + x_2$ is odd, we analogously get a contradiction. □

2.1.4. Two cycles of length 3. There are two permutations with a cycle of length 3,

$$\begin{cases} \phi_0: & \boldsymbol{i} \to \boldsymbol{j} \to \boldsymbol{k} \to \boldsymbol{i} \quad \text{and} \quad -\boldsymbol{i} \to -\boldsymbol{k} \to -\boldsymbol{j} \to -\boldsymbol{i} \\ \phi_1: & \boldsymbol{i} \to \boldsymbol{j} \to \boldsymbol{k} \to \boldsymbol{i} \quad \text{and} \quad -\boldsymbol{i} \to -\boldsymbol{k} \to -\boldsymbol{j} \to -\boldsymbol{i} \end{cases}$$

and

$$\begin{cases} \phi_0': & \boldsymbol{i} \to \boldsymbol{j} \to \boldsymbol{k} \to \boldsymbol{i} \quad \text{and} \quad -\boldsymbol{i} \to -\boldsymbol{j} \to -\boldsymbol{k} \to -\boldsymbol{i} \\ \phi_1': & \boldsymbol{i} \to \boldsymbol{k} \to \boldsymbol{j} \to \boldsymbol{i} \quad \text{and} \quad -\boldsymbol{i} \to -\boldsymbol{k} \to -\boldsymbol{j} \to -\boldsymbol{i}. \end{cases}$$

The first rule is trivial since both rules are identical. The second rule is equally simple. If the ant starts with a velocity in the set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, its velocity will remain in this set forever, making the ant propagate without passing more than once through each cell. Its trace is included in the cone $\mathbb{Z}^3_+$.

2.1.5. Three cycles of length 2. There is only one permutation with three cycles of length 2,

$$\begin{cases} p_0: & \boldsymbol{i} \leftrightarrow \boldsymbol{j}, \quad \boldsymbol{k} \leftrightarrow -\boldsymbol{i} \quad \text{and} \quad -\boldsymbol{j} \leftrightarrow -\boldsymbol{k} \\ p_1: & -\boldsymbol{i} \leftrightarrow -\boldsymbol{j}, \quad -\boldsymbol{k} \leftrightarrow \boldsymbol{i} \quad \text{and} \quad \boldsymbol{j} \leftrightarrow \boldsymbol{k}. \end{cases}$$

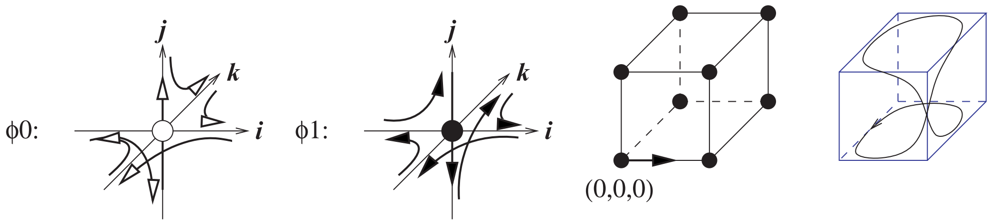


Figure 5. A rule with a straight fragment, an initial colouration and the produced trace. The initial position is $(0, 0, 0)$ and the initial velocity is $-\boldsymbol{k}$. The period is 16.

Like rule (h_0, h_1), this ant is equivalent to the ant over the hexagonal grid. In fact, the set of velocities can be divided in two sets: $\{\boldsymbol{i}, -\boldsymbol{j}, \boldsymbol{k}\}$ and $\{-\boldsymbol{i}, \boldsymbol{j}, -\boldsymbol{k}\}$, from what we can prove the analogous of lemma 1 and theorem 1.

2.2. General comments

In [7], it is proved that *if the ant's trajectory is periodic for some initial colouration and the transition rule has a scattering rule without backward fragments, then the scattering rule cannot contain a straight fragment*. This is a strong property of two-dimensional Lorentz's lattice gases which says that only rotators allow periodic trajectories. But a similar result is not true in dimension 3, as the following counterexample shows.

Let us consider the following rule:

$$\begin{cases} \phi_0: & \boldsymbol{j} \to -\boldsymbol{k} \to -\boldsymbol{i} \to \boldsymbol{k} \to \boldsymbol{i} \to \boldsymbol{j} \qquad \text{and} \qquad -\boldsymbol{j} \to -\boldsymbol{j} \\ \phi_1: & -\boldsymbol{j} \to -\boldsymbol{k} \to -\boldsymbol{i} \to \boldsymbol{k} \to \boldsymbol{i} \to -\boldsymbol{j} \qquad \text{and} \qquad \boldsymbol{j} \to \boldsymbol{j}. \end{cases}$$

It contains a straight fragment $(-\boldsymbol{j}, -\boldsymbol{j})$, and it allows a periodic trajectory. Figure 5 shows the diagram of the rule and the initial colouration that produces a periodic trajectory. The straight fragment is never used. The rule is not time symmetric.

Within the three-dimensional rules, we have found some rules that produced traces that are trapped on a two-dimensional subspace, and some of these rules resulted to be equivalent to some already known versions of Langton's ant. This will happen on larger dimensions too. In fact, taken the ideas of section 2.1.2, any rule defined on dimension n can be embedded on dimension $n+1$ by transforming the vector e_n by $e_n + e_{n+1}$. More precisely, if a transition rule (ϕ_0, ϕ_1) is defined on $\mathbb{Z}^n$, then we define the rule (ϕ_0', ϕ_1') on $\mathbb{Z}^{n+1}$ by: $\phi_i'(e_n) = e_{n+1}$ and $\phi_i'(-e_{n+1}) = -e_n$ for $i = 0$ and $i = 1$, and $\phi_i'(e_{n+1}) = \phi_i(e_n)$, $\phi_i'(\phi^{-1}(-e_n)) = -e_{n+1}$, and $\phi'(v) = \phi(v)$ if v is not comprised by the former definitions. Analogously, any rule on dimension k can be embedded on dimension n if $k \leqslant n$.

Lemma 1 can be generalized to every odd dimension, if we generalize the corresponding rule accordingly. This is achieved by the transformation F defined by (R, R^T), where R is represented by the following matrix:

$$R = \begin{pmatrix} 0_{1\times 2n} & -1 \\ I_{2n} & 0_{2n\times 1} \end{pmatrix}.$$

As lemma 1, we can prove that if $(c, x, \boldsymbol{v}) = F^t(c_0, x_0, \boldsymbol{v_0})$, then

$$\left(x + \tfrac{1}{2}\boldsymbol{v}\right) \cdot (1 \quad -1 \quad 1 \quad -1 \quad \cdots \quad 1)^T = C_0,$$

for some constant C_0. This proves that the ant is always alternating between two hyperplanes. The system is then equivalent to another system on dimension $2n$.

Table 1. If all the vertices have colour 0, only rule 0 will be applied within the first four steps. The table shows the evolution of the ant's basis and position.

	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
x	(0, 0, 0)	(1, 0, 0)	(1, 1, 0)	(1, 1, 1)	(0, 1, 1)
b^1	(1, 0, 0)	(0, 1, 0)	(0, 0, 1)	(−1, 0, 0)	(0, −1, 0)
b^2	(0, 1, 0)	(0, 0, 1)	(−1, 0, 0)	(0, −1, 0)	(0, 0, −1)
b^3	(0, 0, 1)	(−1, 0, 0)	(0, −1, 0)	(0, 0, −1)	(1, 0, 0)

3. Computer science approach

From the computer science approach, the ant is viewed as a Turing machine whose tape is a n-dimensional grid. Its interest comes from artificial life. The ant is not supposed to have any knowledge about a global coordinate system, it can only distinguish directions relative to itself (up, left, etc). On a white vertex, for example, it can decide to go up and to rotate its back to the left; this corresponds to the transformation: *ahead:=up, up:=left, left:=ahead*. It will always move like this on white vertices, independently on which is its current *ahead* direction. The ant's state should be given by an orthogonal basis, and its transition function should be defined as follows.

Definition 5 (the ant of computer scientists)**.** *A movement rule is defined as a permutation* $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ *together with a sign assignment* $\sigma : \{1, \ldots, n\} \to \{+1, -1\}$.

Let the set of colours be $C = \{L, R\}$. *A transition rule is a pair of movement rules* $((\pi_L, \sigma_L), (\pi_R, \sigma_R))$.

The state set Q *is the set of orthogonal basis of* $\mathbb{Z}^n$ *composed by vectors of* $\mathcal{D}$, *i.e.,* $Q = \{(u^1, u^2, \ldots, u^n) \in \mathcal{D}^n | (u^1, u^2, \ldots, u^n)$ *is a basis of* $\mathbb{Z}^n\}$.

The dynamics of the ant transforms a state $(c, x, u^1, \ldots, u^n)$ *of the system into another state* $(c', x', u'^1, \ldots, u'^n)$ *as follows:*

- $x' = x + u^1$
- $u'^i = \sigma_{c(x')}(i) u^{\pi_{c(x')}(i)}, \ i \in \{1, \ldots, n\}$
- $c'(x') \neq c(x')$
- $\forall y \neq x', c'(y) = c(y)$.

For example, if $\pi_0 = (2, 3, 1), \sigma_0 = (+1, +1, -1)$, then the position and basis of the ant will evolve as table 1 shows.

With this definition, the ant is made to have only a relative knowledge of its background. The basis represents the 'body' of the ant. A similar work has been made by Hamann [13] in $\mathbb{Z}^3$. He described the ant's state by a vector in $\mathcal{D}$ together with an oriented plane. He considered only four possible movements: *right, left, up* and *down*, which correspond to the rules: $((3, 2, 1), (-1, 1, 1))$, $((3, 2, 1), (1, 1, -1))$, $((2, 1, 3), (1, -1, 1))$ and $((2, 1, 3), (-1, 1, 1))$ in the present framework. With these rules, a set of only two colours is not enough for having interesting rules, as he remarked. In fact, if both rules are of the same kind, *(right, left)* or (*up, down*), then the ant's behaviour is two dimensional. If the two rules are of different kinds, for example, (*right, down*) or (*up, right*), the trace is restricted to a cube of side length 2 for every initial configuration. Hamann showed simulations of all the rules with three and four colours and an homogeneous initial configuration. He found many rules that occasionally produced a regular movement. He reported also several rules behaving in two dimensions or trapped into a set of less than eight vertices.

We will restrict our attention to the simplest case, where $\pi_R = \pi_L$. In this case, we say that the rule is *space symmetric*. With this restriction we will be able to characterize equivalent transition rules and to prove strong results about the ant's dynamics which are presented in section 3.1. This case is of particular importance to us because Langton's ant is space symmetric.

A coordinate basis has an orientation. Some of our movement rules change the basis orientation, some not. In section 3.2 we show how the notion of orientation has a particular interpretation in the present context, and we illustrate this with some examples in dimensions 1, 2 and 3.

Section 3.3 is devoted to the three-dimensional case, where only seven non-equivalent transition rules have been found. One of these rules is proved to be Turing universal.

3.1. Space symmetric rules in the n-dimensional lattice $\mathbb{Z}^n$

If a transition rule is space symmetric, it is of the form: $((\pi, \sigma_L), (\pi, \sigma_R))$. Following definition 5, we observe that if the ant starts with the basis $(u^1, \ldots, u^n)$, its first movement will be parallel to the vector u^1, its second movement will be parallel to $u^{\pi(1)}$, its third movement will be parallel to $u^{\pi^2(1)}$ and so on, until the ant recovers its initial direction (up to a sign) $u^1 = u^{\pi^k(1)}$, for some k. This is expressed by the following proposition which can be easily proved by induction.

Proposition 1. *Let $((\pi, \sigma_R), (\pi, \sigma_L))$ be a transition rule in $\mathbb{Z}^n$, and let $\left(c_t, x_t, u_t^1, \ldots, u_t^n\right)$ be the configuration of the system at time t, for each $t \geqslant 0$. Then it holds that*

$$(\forall t, i \geqslant 0) u_t^{\pi^i(1)} = \left(\prod_{j=1}^{t} \sigma_{c_{j-1}(x_j)}(\pi^{i+t-j}(1)) \right) u_0^{\pi^{i+t}(1)}.$$

Definition 6. *Given a permutation π, we call principal length of π the smallest positive integer k such that $\pi^k(1) = 1$. We call principal cycle the set $\mathcal{O}(\pi) = \{\pi^i(1)\}_{i=0}^{k-1}$.*

The ant will be restricted to the space generated by the vectors whose indices are in the principal cycle of π, and it will follow these vectors in a cyclic way. The following proposition states that we can partition the space into $2k$ sets such that the ant's position changes from one set to the other in a cyclic way. This proves that the support graph is $2k$ partite and it is contained in a k-dimensional subspace.

Proposition 2. *Let $((\pi, \sigma_R), (\pi, \sigma_L))$ be a transition rule in $\mathbb{Z}^n$, and let k be the principal length of π. Then the support graph of the system is $2k$-partite, directed, of out-degree at most two, and if the initial position is $x_0 = (0, 0, \ldots, 0)$ and the initial basis is $(u^1, u^2, \ldots, u^n)$, then the support graph is contained in the subspace generated by the vectors $\{u^1, u^{\pi(1)}, \ldots, u^{\pi^{k-1}(1)}\}$.*

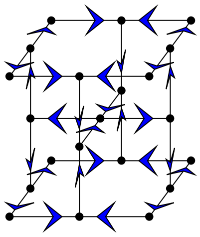
Proof. Let us define the $2k$-partition $(S_j)_{j=0}^{2k-1}$ as follows, for each j such that $0 \leqslant j \leqslant k-1$, we set

$$\begin{cases} S_j = \left\{ \sum_{i=1}^{k} \alpha_i u^{\pi^{i-1}(1)} : (\forall i \leqslant j)\ \alpha_i \text{ is odd } \wedge (\forall i > j)\ \alpha_i \text{ is even} \right\} \\ S_{j+k} = \left\{ \sum_{i=1}^{k} \alpha_i u^{\pi^{i-1}(1)} : (\forall i \leqslant j)\ \alpha_i \text{ is even } \wedge (\forall i > j)\ \alpha_i \text{ is odd} \right\} \end{cases}.$$

The result is obtained by proving, by induction on the number of iterations, that at iteration $2kq + d$ (with $0 \leqslant d < 2k$) the ant is in S_d. □

3	2	3	2
0	1	0	1
3	2	3	2
0	1	0	1

(*a*)



(*b*)

Figure 6. (*a*) The 4-partition for the two-dimensional case, with $k = 2$. (*b*) A part of the support graph for $k = 3$.

Figure 6 shows the $2k$-partition for $k = 2$ and the initial basis $(u^1, u^2) = (\vec{i}, \vec{j})$, and a sketch of the support graph for $k = 3$. We remark that for $n \geqslant 3$ and $k = n$, not every vertex of $\mathbb{Z}^n$ is in the support graph. For example, within the cube $\{0, 1\}^n$ only $2n$ vertices belong to the support graph. If $k < n$, the trace is restricted to a k-dimensional subspace. It is not difficult to see that such a rule is in some sense equivalent to some rule defined on $\mathbb{Z}^k$.

The following theorem establishes sufficient conditions for two rules in the same space to be equivalent.

Theorem 3. *Let* $((\pi_1, \sigma_{R1}), (\pi_1, \sigma_{L1}))$ *and* $((\pi_2, \sigma_{R2}), (\pi_2, \sigma_{L2}))$ *be two transition rules defined in* $\mathbb{Z}^n$ *with principal lengths equal to* $k \leqslant n$. *If the sign vectors satisfy that:*

(i) for every $j \in \{0, \ldots, k-1\}$, $\sigma_{L1}\big(\pi_1^j(1)\big)\sigma_{L2}\big(\pi_2^j(1)\big) = \sigma_{R1}\big(\pi_1^j(1)\big)\sigma_{R2}\big(\pi_2^j(1)\big)$, *and*

(ii) there exists a $2k$ *periodic function* $f : \mathbb{N} \to \{L, R\}$ *such that*

$$(\forall t \in \{0, \ldots, 2k-1\}) \prod_{j=t}^{t+k-1} \sigma_{L1}\big(\pi_1^{t-j}(1)\big)\sigma_{f(j)2}\big(\pi_2^{t-j}(1)\big) = 1.$$

Then, there is a transformation Ψ *such that for every state* $(c, x, u^1, \ldots, u^n)$, $\mathcal{T}_1(c, x, u^1, \ldots, u^n) = \mathcal{T}_2(\Psi(c, x, u^1, \ldots, u^n))$.

Proof. Let us define Ψ by $\Psi\big(c_0, x_0, u_0^1, \ldots, u_0^n\big) = \big(\tilde{c}_0, x_0, v_0^1, \ldots, v_0^n\big)$, where

$$v_0^l = s(l)u_0^{p(l)}, \qquad l \in \{1, \ldots, n\},$$

where s is a sign assignment and p is a permutation. In order to have equal traces in both systems, we need $u_t^1 = v_t^1$ for every t. Using proposition 1, this gives the following equations:

$$(\forall t \in \mathbb{N}) p\big(\pi_2^t(1)\big) = \pi_1^t(1), \qquad \text{and} \tag{1}$$

$$(\forall t \in \mathbb{N}) s\big(\pi_2^t(1)\big) = \prod_{j=1}^{t} \sigma_{c_{j-1}(x_j)1}\big(\pi_1^{t-j}(1)\big)\sigma_{\tilde{c}_{j-1}(x_j)2}\big(\pi_2^{t-j}(1)\big). \tag{2}$$

Equation (1) defines p on $\mathcal{O}(\pi^2)$. Equation (2) defines s on $\mathcal{O}(\pi^2)$, but it is also required that $s\big(\pi_2^t(1)\big) = s\big(\pi_2^{t+k}(1)\big)$ for every t. We only need to define $\tilde{c}_0$ appropriately; condition (ii) will be used to achieve this.

Let us first remark that condition (i) implies that $\sigma_{c_j(x)1}\big(\pi_1^i(1)\big)\sigma_{\tilde{c}_j(x)2}\big(\pi_2^i(1)\big)$ do not depend directly on c_j, but may take two values depending on whether $c_j(x)$ and $\tilde{c}_j(x)$ are equal or different. More precisely, we have that

$$\sigma_{c_j(x)1}\left(\pi_1^i(1)\right)\sigma_{\tilde{c}_j(x)2}\big(\pi_2^i(1)\big) = \begin{cases} \sigma_{L1}\big(\pi_1^i(1)\big)\sigma_{L2}\big(\pi_2^i(1)\big) & \text{if} \quad c_j(x) = \tilde{c}_j(x) \\ \sigma_{L1}\big(\pi_1^i(1)\big)\sigma_{R2}\big(\pi_2^i(1)\big) & \text{if} \quad c_j(x) \neq \tilde{c}_j(x). \end{cases}$$

Figure 7. A simulation of two equivalent rules in dimension 2.

The idea is to define $\tilde{c}_0$ in such a way that

$$\sigma_{c_j(x_j)1}\left(\pi_1^i(1)\right)\sigma_{\tilde{c}_j(x_j)2}\left(\pi_2^i(1)\right)=\sigma_{L1}\left(\pi_1^i(1)\right)\sigma_{f(j)2}\left(\pi_2^i(1)\right).$$

On the other hand, taking the notation from proposition 2, we know that $x_j \in S_i + \{x_0\}$ if and only if $j - i$ is a multiple of $2k$. Then we define $\tilde{c}_0(x)$ to be the opposite of $c_0(x)$ for every $x \in S_i$ such that $f(i) = R$, and equal to $c_0(x)$ in the other case.

Then s and p are well defined on $\mathcal{O}(\pi)$. We do not care about the values of s and p outside $\mathcal{O}(\pi_2)$. They can be defined arbitrarily.

With this definition, the velocities are equal in both systems and, since the initial positions are also equal, both traces are equal. □

This theorem applies to the rules $(\pi_1 = (2,1), \sigma_{L1} = (-1,1), \sigma_{R1} = (1,-1))$ and $(\pi_2 = (2,1), \sigma_{L2} = (1,1), \sigma_{R2} = (-1,-1))$. The function $f(2t) = L$ and $f(2t+1) = R$ satisfies the hypothesis. Figure 7 shows the configurations of both systems at time 10 837. We can see how the transformation Ψ changes the configuration according to f.

3.2. The concept of basis orientation

A pertinent restriction may be to consider only movement rules that preserve the orientation of the basis. From geometry (see, for example, [16]), two basis have the same orientation if the matrix that transforms one to the other has a positive determinant. In the present case, the determinant of the matrix for a given movement rule (π, σ) is equal to $\mathrm{sign}(\pi) \times \prod_{i=1}^{n} \sigma(i)$. Thus, we will consider only movement rules such that

$$\mathrm{sign}(\pi) = \prod_{i=1}^{n} \sigma(i). \tag{3}$$

In dimension 1, this implies that $\sigma = (1)$, which expresses the fact that an insect cannot turn back in dimension 1.

In dimension 2, the analysis depends on the principal length of π. If it is a cycle of length 1 (i.e., $\pi = (1,2)$), the product $\sigma(1)\sigma(2)$ must be 1, but only $\sigma(1)$ is relevant for the ant's trace, and it can have any sign. This is interpreted by the fact that, in dimension 2, an insect can turn back. Figure 8 shows a simulation of a rule defined on dimension two, whose movement is restricted to dimension one. The ant's body is represented by an arrow and we can see that it can turn back.

On the other hand, if the principal length is 2, there are only two possible sign functions $(1,-1)$ and $(-1,1)$, then there is only one transition rule: Langton's ant's rule. This expresses the fact that if the ant is 'over' the plane, it cannot go 'under' the plane without passing by a third dimension. But if we are in dimension 3, with a rule with principal length 2, passing

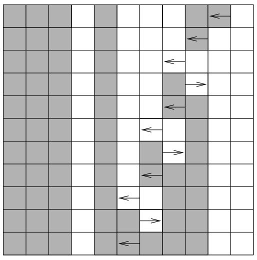


Figure 8. A simulation of a rule whose movement is restricted to a one-dimensional subspace. We show the evolution of the system in time, each row representing the state of the system at a different iteration. The time progresses downwards.

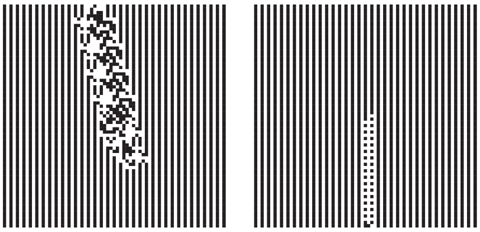


Figure 9. A simulation of two rules defined in dimension 3, but that behave in dimension 2.

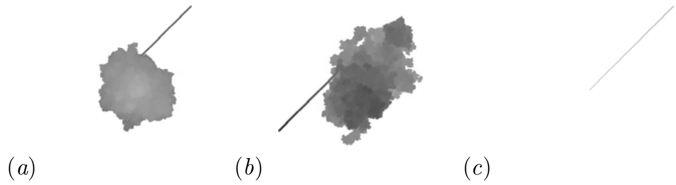


Figure 10. Simulations of the three rules with principal length 3. The three rules have $\pi = (3, 1, 2)$ and $\sigma_L = (1, 1, 1)$. The figure shows the points visited by the ant when starting over a given periodical colouration. The rules are: (*a*) $\sigma_R = (1, -1, -1)$, (*b*) $\sigma_R = (-1, 1, -1)$ and (*c*) $\sigma_R = (-1, -1, 1)$.

under the plane is possible, and three new transition rules appear, one of them being equivalent to Langton's ant. Figure 9 shows simulation of these two transition rules for an initial ribbed colouration.

3.3. The three-dimensional case

If we consider only symmetric transition rules which respect the basis orientation, seven non-equivalent rules are left. We have already presented four of them. The other three correspond to rules with a permutation with principal cycle of length 3: $\pi = (3, 1, 2)$. Figure 10 shows simulations of these rules over an initial periodic colouration. The behaviour of two of them seems difficult to predict. In fact, they are Turing universal. The proof uses the same ideas on the proof of universality of two-dimensional Langton's ant in [10].

The idea is to represent Boolean formulae through initial colourations defined on a box. Figure 11 represents a box that can compute an AND gate. When the ant enters, we make

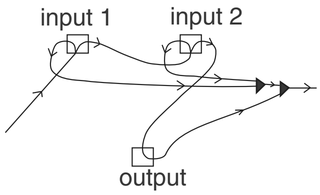


Figure 11. A scheme of an AND gate. The ant follows the paths and bifurcates depending on the colour of the vertices labelled 'input 1' and 'input 2'. The triangles represent the junction of two paths.

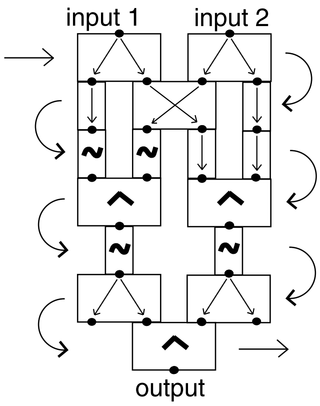


Figure 12. The XOR formula. The small rectangles represent basic gates. The ant enters and exits the successive layers of gates to finally compute the value of the complete formula.

it to visit some given vertices labelled as 'input vertices'. Depending on the colour of these vertices, the ant will follow different traces. Then, we can drive the ant either to an exit or to some 'output vertices', where the ant 'writes' the output of the Boolean formula. The 'output vertices' start always with white colour; then if the ant visit one of them, it changes its colour and exits by a given fixed arc. The 'output vertex' becomes black only if the ant visits it. In this way, the ant 'writes' the output.

Other Boolean formula can be computed by successively stacking basic gates as AND, NOT, COPY and FANOUT. The 'output vertices' of some gates are used as 'input vertices' of others. The ant enters and exits the successive layers of gates and finally computes the value of the complete formula. The construction of the XOR formula is shown in figure 12.

In order to implement this idea, we need the ability of conducting the ant from any vertex to another, and we need to be able to make two different traces to join into a single one. It is not difficult to see that we may have the ant go from a chosen vertex to another in the support graph by choosing an appropriate initial colouration. It is less easy to see that we can also choose the coordinate basis of the ant at the initial and final positions. Figure 13(a) shows how to make two different basis change into a chosen one for the rule $(\pi = (3, 1, 2), \sigma_L = (1, 1, 1), \sigma_R = (1, -1, -1))$. The two basis are the only two possible ones for a given fixed initial basis. The second task is done by defining a configuration that can join two different traces. Figure 13(b) shows this configuration. The ant can enter by vertices $e1$ and $e2$, and always exit by the same vertex.

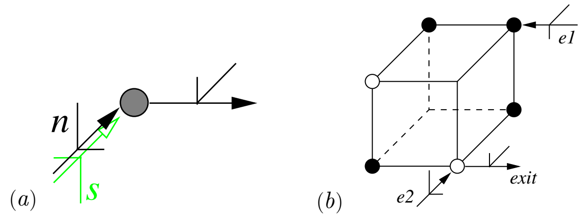


Figure 13. (*a*) If the ant enters with a basis n, we colour the vertex white, and if the basis is s, we choose black colour; in this way, we make the ant exit with the indicated basis. (*b*) The ant can enter by vertices $e1$ and $e2$, and always exit by the same vertex.

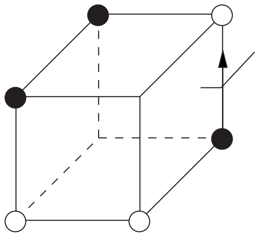


Figure 14. With this initial colouration, the ant with rule $\sigma_R = (-1, -1, 1)$ stays inside this cube, producing a dynamic of period 12.

We conclude that the rule $(\pi = (3, 1, 2), \sigma_L = (1, 1, 1), \sigma_R = (1, -1, -1))$ is Turing universal. A similar work can be done for the rule $(\pi = (3, 1, 2), \sigma_L = (1, 1, 1), \sigma_R = (-1, 1, -1))$.

The existence of periodic trajectories for these rules is an open problem. We only found a colouration that makes rule $(\pi = (3, 1, 2), \sigma_L = (1, 1, 1), \sigma_R = (-1, -1, 1))$ produce a periodic trajectory. This colouration is shown in figure 14. We did not find an analogous configuration for the other two rules.

4. Final comments and open problems

We have defined two formulations of the ant in an n-dimensional lattice. These formulations have physical interpretations and, at the same time, are practical for theoretical analysis. They also are easily adaptable if the lattice corresponds to another algebraic structure. For example, if the structure is a finitely generated group, it is enough to replace the velocities set $\mathcal{D}$ by the group generators and their inverses. Some of our results should be easy to extend to this case, at least for some particular groups.

In order to study the system, we have defined the notion of *support graph* which represents the vertices and arcs that can be reached by the ant for a given fixed initial position. This tool proved to be very useful to describe the properties of a system and to simplify the proofs.

In dimension 3, the simplest rules were studied in detail. We would like to stress some facts about these rules. First, several rules were proved to produce traces which are contained in a one- or two-dimensional sublattice. This remark reinforces a Bunimovich [12] result. Bunimovich also proved a similar result for dimension n. It also seems that every rule in dimension n that admits finite traces has a support graph that can be embedded in a $(n-1)$-dimensional space, this is still to be proved.

Several variations of Langton's ant have been studied on a two-dimensional lattice. Among them are the static model where the vertices do not change their colour, or the multi-ant system. Our results about the support graph are still valid in these two cases, because they lay on the geometric properties of the transition rule. Other important variations are: (*a*) the non-full model, where some vertices do not act on the ant (are empty) or (*b*) more than two colours are admitted. Our results cannot be extended to these cases.

The present paper shows some results on the support graph of the system, but it says little about the dynamics. Important open questions arise: for a given rule, is the trace always infinite? Which are the diffusion types of these rules?

We did not study the non-symmetrical case, i.e., when the permutation associated with each colour is different. From computer science and artificial life, space symmetry is probably not an important property. Non-symmetric rules should therefore also be studied.

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